Euclid's Kinds and (Their) Attributes

Benjamin Wilck

Department of Philosophy, Faculty of Philosophy, Humboldt-Universität,
Berlin, Germany

benjamin.wilck@hu-berlin.de

Abstract

Relying upon a very close reading of all of the definitions given in Euclid's *Elements*, I argue that this mathematical treatise contains a philosophical treatment of mathematical objects. Specifically, I show that Euclid draws elaborate metaphysical distinctions between (i) substances and non-substantial attributes of substances, (ii) different kinds of substance, and (iii) different kinds of non-substance. While the general metaphysical theory adopted in the *Elements* resembles that of Aristotle in many respects, Euclid does not employ Aristotle's terminology, or indeed, any philosophical terminology at all. Instead, Euclid systematically uses different types of definition to distinguish between metaphysically different kinds of mathematical object.

Keywords

Euclid's *Elements* – mathematical objects – definitions – method of division – Aristotle – substance – genus – differentia

1 Introduction

Euclid's *Elements*, c. 300 BCE,¹ is the earliest extant treatise of deductive mathematics in history, and is still regarded as a paradigm of an axiomatic science.² By deriving a large number of mathematical theorems from a relatively small

1 On the question of Euclid's dating, see Proclus, in *Eucl.* 68, 10–20; cf. Heiberg 1882, 25–26; Heath 1925, i, 1–2.

2 Frege 1893, VI; Hilbert 1917, 407–408; Tarski 1937, 80, n. 2; Eves 1990, 13, 26; Linnebo 2017, 21–24.

number of undemonstrated principles (definitions, postulates, common notions), the *Elements* seems to be a purely mathematical treatise. In the present paper, however, I show that the *Elements* also conveys a metaphysical theory of mathematical objects. Specifically, I argue that Euclid promotes elaborate metaphysical distinctions between

- (i) substances and non-substantial attributes of substances;³
- (ii) different kinds of substance;
- (iii) different kinds of non-substance.

Pertaining to the subdivision of substances, I argue that Euclid distinguishes between generic and specific substances, and that he further subdivides specific substances into divisible and indivisible substances. Pertaining to the subdivision of non-substances, I argue that Euclid distinguishes between differentiae and relatives. (These Aristotelian notions will be elucidated in the course of this paper.) This yields the following ontological classification of Euclid’s mathematical universe:

The possibility that Euclid’s definitions convey a metaphysical theory of mathematical objects has been either ignored by scholarship on Euclid and ancient philosophy, or has been rejected as false. Maziarz and Greenwood (1968, 243), for instance, believe that “the definitions given in the *Elements* are merely asserted or proved, without giving any indication as to the nature of the objects defined.”

3 While Euclidean non-substances are non-substantial attributes of Euclidean substances, Euclidean substances too are in fact a certain kind of attribute, namely substantial attributes; hence the pronoun ‘their’ in the title of the present paper.

The very idea of holding Euclid to draw metaphysical distinctions between various kinds of mathematical attribute might strike the reader as surprising, and understandably so, given that Euclid at no point makes any assertions about the ontological status of the mathematical objects he studies. The *Elements'* definitions might seem rather innocent as regards their metaphysical commitments. Nothing in these definitions seems to indicate that Euclid draws any kind of metaphysical distinctions.

Or so it seems. A thorough analysis of the formulations of Euclid's definitions reveals that Euclid draws the aforementioned metaphysical distinctions by way of subtle linguistic encryptions. Euclid in fact systematically uses different types of definition to distinguish between metaphysically different kinds of mathematical object. He thereby, without explicitly formulating a metaphysical theory, encodes a specific metaphysical theory into his mathematical text.

In this paper, I am chiefly concerned with identifying and interpreting the structural differences that Euclid uses in formulating his definitions, arguing that they serve as markers of Euclid's metaphysical distinctions. My paper thereby provides the first self-contained study of Euclid's method of definition, relying upon a very close reading of all of the definitions given in the *Elements*.

The present study shows that seemingly innocuous differences in Euclid's formulations of definitions have substantial philosophical import. The linguistic regularities found in his definitions have unexpected consequences for interpreting Euclid's overall project. In addition, these results even warrant revisions of Heiberg's edition of the original text of the *Elements*, as I will show with the case of Euclid's infamous definition of parallel lines (*El.* I, Def. 23, see Section 5.4 below).⁴

By saying that Euclid distinguishes mathematical substances from mathematical non-substances, I do not mean to claim that Euclid regards some mathematical objects, such as numbers or triangles, as substances in the sense of being the constituent material or formal elements of physical objects. Rather, I would like to argue that Euclid's metaphysical distinctions are in

4 I consider all definitions that have been transmitted by the manuscripts and are listed in Heiberg's edition of the *Elements*, including Euclid's alternative definitions, as well as those which are most likely inauthentic (some deviations from the linguistic patterns analyzed below can be explained as being the results of editorial interpolations). I reference Euclid's definitions as follows: Roman numerals refer to the *Elements'* book numbers, while Arabic numerals refer to particular definitions within a book. Where several definitions are listed under one single Arabic numeral, I use i, ii, iii, ... to introduce subordinate enumerations. I follow the enumeration established by Heiberg 1883–1876, which is also followed by Heath (1925), Mueller (1981, 317–370), and Vitrac (1990–2001). (Heath's enumeration diverges in *El.* VII since he dismisses *El.* VII, Def. 10). English translations from the Greek are my own.

line with Aristotle's logical treatment of some mathematical objects as quasi-substances. While Aristotle argues that mathematical objects are nothing but non-substantial attributes of concrete physical substances,⁵ this does not prevent Aristotle from treating some mathematical objects as quasi-substances, analogously to physical substances. For instance, Aristotle treats 'line' as the primary substantial subject of non-substantial attributes such as 'straight', as well as 'number' as the primary substantial subject of non-substantial attributes such as 'even' and 'odd'.⁶ As we shall see, Euclid follows Aristotle in this.

Still, Euclid does not employ Aristotle's terminology, or indeed, any philosophical terminology at all.⁷ Metaphysical terms such as 'substance', 'genus', 'species' (or 'kind'), 'differentia', and 'relative' are entirely absent in Euclid's writings. (Remarkably, however, the *Elements* features the term 'definition' (*horos*) in the headings of the lists of Euclid's definitions at the beginnings of Books I–VII and X–XI.⁸) The absence of a philosophical vocabulary in the

5 *Metaphysics* (= *Metaph.*) 1077b11–30.

6 *Metaph.* 1031a2–4.

7 Apart from some second-order discourse such as personal language, headings, and occasional back-references, the *Elements* features only scarce occurrences of metamathematical vocabulary, such as forms of 'to prove' (*deiknumi*) and 'to be assumed' (*hupokeinai*). Netz (1999, 94–103) takes all of Euclid's definitions to belong to second-order discourse. While my argument in the present paper indeed supports the view that Euclid's definitions convey extra-mathematical views about the mathematical objects defined, Euclid's definitions still belong to first-order discourse insofar as Euclid expressly appeals to many of his definitions as deductive premises in subsequent mathematical proofs.

8 While *horos* occurs only eleven times in the entire *Elements*, it is used in three different meanings. Besides eight occurrences of *horos* in the sense of 'definition', *horos* occurs twice in the sense of 'boundary' in Euclid's definitions of 'boundary' and 'figure' (*El.* I, Def. 13–14): "A boundary (*horos*) is that which is the limit (*peras*) of something. A figure (*schēma*) is that which is contained by a certain boundary (*horos*) or certain boundaries (*horoi*).". Moreover, the word *horos* occurs once in the sense of 'term of a proportion', namely in *El.* v, Def. 8: "A proportion (*analogia*) is in three terms (*horoi*) at the least". These three different senses of *horos* are in fact closely connected. The boundary of a certain figure does not only bound that figure, but it is also mentioned in its definition. For example, the triangle is not only bounded by straight lines, but also defined by the terms 'straight' and 'line' (cf. Euclid's definition of the triangle as "a rectilinear figure contained by three straight lines", *El.* I, Def. 19ii). Also, proportions are both bounded and defined by their terms. In general, then, the boundaries by which a mathematical object is contained are contained in its definition. Aristotle too uses *horos* in the sense of 'definition', 'boundary', 'term of a proportion', but also in the sense of 'term of a proposition'. (Terms are called *horoi* because propositions are both bounded and defined (*horizontai*) by terms: cf. Alexander of Aphrodisias, *On Aristotle Prior Analytics* 14,29–15,1.) Moreover, Aristotle appears to play with the ambiguity of the term *horos* when he says that the point is the *horos* (boundary / definition) of the line, the line that of the surface, and the surface that of the solid (see, e.g., *Metaph.* 1002a4–8; 1017b17–21; *Physics* (= *Phys.*) 249a25–27). Accordingly, Frege (1884, §§ 26; 88; 1903, §§ 56–65), too, conceives of definitions

Elements calls for the introduction of a suitable terminology to capture Euclid's metaphysical distinctions. Since Euclid's metaphysical theory can be found to resemble Aristotle's corresponding theory in many respects, I conveniently map Aristotle's vocabulary onto Euclid's distinctions. This is not supposed to suggest that Euclid was an Aristotelian philosopher. Rather, this is a speculative attempt to make sense of the fact that Euclid systematically distinguishes between different kinds of definiendum.⁹

In the course of the present paper, I will emphasize similarities between Euclid's distinctions and Aristotle's metaphysical theory, while also pointing out several divergences between Euclid's mathematical practice and Aristotle's theory of definition.

In view of the fact that the metaphysical theory found in Euclid's *Elements* echoes related theories in Aristotle and the Platonic Academy, the *Elements* turns out to be an early instance of the history of the reception of ancient philosophy in the Aristotelian and Platonic tradition. Still, the study of Euclid's definitions and their philosophical implications is equally interesting in its own right. The present inquiry offers a new perspective on a classical text, not only for historians of philosophy, but also for contemporary philosophers working on the theory of essence or philosophy of mathematics. Moreover, since the *Elements'* underlying metaphysical theory of definition has not been examined to date, the present study fills—or, rather, opens—a research lacuna in the existing scholarship on Euclid and the history of ancient mathematics and science. Furthermore, insofar as this paper shows that the *Elements* encodes metaphysical distinctions by certain linguistic encryptions, it is of interest for philological scholarship concerned with the literary strategies of ancient technical texts.

2 Methodological Remarks

Already in late antiquity, attempts were made to understand Euclid from a philosophical perspective. In the fifth century CE, for example, Proclus argued that the *Elements* is a cosmological treatise about the geometrical elements of the physical world because the *Elements* culminates in the construction of

as sharp boundaries, namely, of concepts. It should be noted, though, that Frege (1903, § 56) takes boundary-talk about definitions and concepts to be merely metaphorical.

9 I use 'definiendum' (plural: 'definienda') for that which is defined, and 'definiens' (plural: 'definientia') for the defining-phrase.

the five regular, Platonic solids, which prominently figure in the cosmology of Plato's *Timaeus*.¹⁰

More recent mathematicians, philosophers, and commentators have engaged in reconstructing Euclid's intuitive conception of three-dimensional space,¹¹ as well as the logical framework of the *Elements*,¹² usually by pointing out deductive gaps and tacit assumptions in Euclid's proofs. While inquiries such as these are insightful in their own right, they are based upon the anachronistic assumption that Euclid's mathematical practice is deficient with respect to their respective authors' mathematical or philosophical objectives.

Fabio Acerbi, by contrast, reconstructs the logic at work in Greek mathematics solely on the basis of linguistic regularities in the texts themselves.¹³ By reconstructing Euclid's metaphysical theory of mathematical objects on the sole basis of grammatical and logical features of Euclid's definitions, my present project is intimately indebted to Acerbi's approach.

One might wonder why there is a philosophical theory encoded in the *Elements*. It is possible that Euclid held metaphysical beliefs about the ontology of mathematical objects. It might also be the case that Euclid felt the need to underpin his mathematical practice by a reputable philosophical theory. Such considerations might seem of minor relevance if we accept the view that the *Elements* is a compilation of different, pre-Euclidean mathematical sources, such as those of the early Platonic Academy, as Proclus suggests.¹⁴ Still, this can hardly explain the regularity and systematicity with which Euclid coordinates different types of definition with different kinds of definiendum.

The safest way to explain the presence of philosophical layers to the text of the *Elements* is to assume that Euclid is adopting a traditional, philosophically motivated standard for formulating definitions. This would suggest that Euclid might not even be aware of the metaphysical theory that is presupposed by that standard. At any rate, one should refrain from psychological speculations concerning Euclid's possible intentions of conveying a philosophical theory. My aim is simply to show that Euclid's practice of definition encodes a specific

10 *in Eucl.* 68.20–25; 70.19–71.5; 71.22–24; 74.11–13. The view that Euclid's *Elements* is actually a theory of the physical world was readopted in the 20th century by Karl Popper (1952, 147–148; 152).

11 See, e.g., Riemann 1854; Russell 1897; Carnap 1922; Dixon 1930; Janich 1992.

12 See, e.g., Frege 1882, 50–51; Pasch 1882; Klein 1926, 203–226; Mueller 1974, 1981; Eves 1990, 37–41. Proclus (*in Eucl.* 214.21–215.8) reports that already the Epicurean philosopher Zeno of Citium complained about deductive gaps in the *Elements*, such as Euclid's tacit assumption that two straight lines cannot have a common segment.

13 See especially Acerbi forthcoming.

14 Proclus, *in Eucl.* 68, 6–10; cf. Knorr 1975, 5–7; 303–313.

philosophical theory, even if we may not be entitled to ascribe this theory to Euclid himself. Thus, when speaking of Euclid's metaphysical distinctions, I do not necessarily refer to Euclid himself. Rather, I refer to the metaphysical theory systematically encoded in Euclid's *Elements*.

Aside from Euclid's practice of drawing structural differences in formulating his definitions, there are other candidates for markers of Euclid's metaphysical distinctions in the *Elements*. These are, most notably, the word form of the definiendum, the lexical choice of the verb connecting the definiendum with the definiens, and the order of definitions. Let me briefly address these three alternative approaches, before then turning to my own analysis of Euclid's definitions.

Pertaining to the word form of the definiendum, Euclid expresses substantial definienda either by:

- (i) a simple noun (such as 'point', *El. I*, Def. 1) or
- (ii) a complex noun phrase (such as 'centre of the circle', *El. I*, Def. 16) or
- (iii) a noun plus an adjective (such as 'trilateral figure', *El. I*, Def. 19ii).

Euclid expresses non-substances either by

- (i) an adjective in combination with a noun (such as 'even number', *El. VI*, Def. 6)¹⁵ or
- (ii) a complex verbal phrase, usually consisting of an infinitive verbal plus a noun (such as 'to multiply a number' *El. VII*, Def. 16).¹⁶

Yet, word form analysis fails to distinguish between cases where Euclid expresses both substances and non-substances by a combination of a noun and an adjective. Moreover, Euclid's various subdivisions of substances and non-substances do not correspond neatly with the word forms of his definienda. Therefore, word form analysis is not suitable to detect Euclid's metaphysical distinctions.

Pertaining to verb choice, Acerbi (2011, 122–123) argues that Euclid distinguishes between one-place predicates and relations, given that in Euclid's *Data*, one-place predicates are defined using the verb form 'is said' (*legetai*), whereas relations are defined using a form of the verb 'to be' (*einai*) or of the verb 'to call' (*kalein*). However, since the *Elements'* definitions do not consistently coordinate different types of definiendum with particular connecting verbs, Euclid's verb vocabulary cannot be found to indicate his metaphysical distinctions in the *Elements*.

15 As I explain in detail in Section 3 below, I am not claiming that Euclid takes even numbers (such as number two) not to be substances, but that he regards the attribute 'even' to be a non-substantial attribute of the genus 'number'.

16 Netz (1999, 91–94) offers a more detailed, but slightly different survey of the word forms of Greek mathematical definienda.

Finally, the order of the *Elements*' definitions is a more promising candidate for detecting Euclid's metaphysical distinctions. Euclid distinguishes metaphysically different kinds of definiendum by conceptual priority relations, given that he systematically defines substances before defining their non-substantial attributes, genera before their species, and wholes before their parts. This appears to follow Aristotle's account of *priority in definition*, according to which *A* is prior in definition to *B* just in case *A* is definable without *B*, whereas *B* is not definable without *A*.¹⁷ The sequence of Euclid's definitions allows us to detect conceptual priority relations not only between different definienda, but also between the definiendum and the terms contained in the definiens. This is indicated by the fact that Euclid systematically defines a term before using it to define other terms (safe for undefined notions). This reflects a crucial requirement for scientific definitions: the terms contained in the definiens must be conceptually more fundamental than the definiendum.¹⁸

Still, the fact that Euclid takes substances to be conceptually prior to non-substances presupposes the identification of what counts as a Euclidean substance and what does not. Moreover, while the ordered sequences of the *Elements*' definitions display several of Euclid's metaphysical distinctions, they do not allow us to retrieve all of those distinctions which are detected by the analysis of different types of definition that I carry out in the present paper.

3 Euclid's Distinction between Substances and Non-Substantial Attributes

Let me begin my reconstruction of Euclid's metaphysical theory of mathematical objects by defending my first main claim: Euclid distinguishes substances from non-substances. This is indicated by the fact that Euclid regularly defines substances by division, but non-substances other than by division.

'Division' (*dihairesis*) is standard Aristotelian vocabulary: the genus is divided by specific differentiae into its different species, and every species is defined by reference to its genus and differentia(e).¹⁹

In terms of numbers, 76 of the 169 *Elements*' definitions are divisive, while 93 are non-divisive. All of the 76 divisive definitions are of substances, while all of the 69 non-substances defined in the *Elements* are defined other than by

¹⁷ *Metaph.* 1018b30–36; cf. 1077b3–4.

¹⁸ I explore the philosophical implications of the order of the *Elements*' definitions in a separate paper.

¹⁹ *Metaph.* 1037b27–1038a35. Cf. *Prior Analytics* (= *APr*) 1.31; *Posterior Analytics* (= *APo*) 11.5; 11.13. Already Plato describes the method of division in *Phaedrus* 265d–266c (cf. *Politicus* 286d–287d), and applies it extensively in the *Sophist*, *Politicus*, and *Philebus*.

division. (5 of the remaining 24 non-divisive definitions are supplements to previously stated definitions, rather than definitions in their own right.²⁰ The other 19 non-divisive definitions are of substances. I examine the latter set of exceptions in Section 3.3 below.)

These rather neat correlations of definitions by division with definitions of substances, and of non-divisive definitions with definitions of non-substances indicate that Euclid systematically distinguishes substances from non-substances.

3.1 *Euclid's First Positive Marker of Substances: Definition by Division*

The fact that Euclid defines only substances by division yields Euclid's first positive marker for detecting his definitions of substances in the *Elements*:

MARK S_{Div}: If *A* is defined by division, then *A* is a substance.

Euclid's definitions of substances can be recognized by the fact that they are defined by division. His definitions by division can in turn be recognized by the fact that they render a genus and a differentia as predicates of the definiendum.

Genus-terms can usually be distinguished from differentia-expressions by their respective word form, as Euclid mostly expresses genera by nouns, but differentiae by adjectives or complex verbal phrases. Given that some of Euclid's genus-terms are complex expressions containing adjectives or verbal phrases, however, the genus-term is best recognized by the fact that it is always stated before the differentia-term.²¹ For instance, in Euclid's definition of 'number', the term 'plurality' is predicated as a genus and the term 'composed of units' as a differentia of the definiendum:²²

A number [is] that plurality which is composed of units.

El. vii, def. 2

20 *El.* I, Def. 3; Def. 6; Def. 17ii; v, Def. 8; xi, Def. 2.

21 This is true even in cases where Euclid places the differentia-term parenthetically between the article and the noun expressing the genus-term, such as in Euclid's definition of 'number' (*El.* I, Def. 2: this definition is quoted in Greek in Section 3.4 below), given that the article of that genus-term is stated first.

22 Aristotle criticizes those who define 'number' as "a composition of units" (*sunthesis monadōn*), arguing that the term 'composition' suggests that one substance—the number—is composed of multiple substances—units—, which Aristotle takes to be impossible (*Metaph.* 1039a4–13). Aristotle himself defines 'number' as "a plurality of units" (*plēthos monadōn*, *Metaph.* 1053a30; 1057a2–3) or "a limited plurality" (*plēthos peperasmenon*, *Metaph.* 1020a13). The fact that Euclid keeps the term 'composed' (*sugkeimenon*) in his definition of 'number' might imply that Euclid's metaphysical account of complex substances diverges from Aristotle's.

Thus, by MARK S_{Div} , the number is a Euclidean substance. Moreover, also Euclid's definition of the line is divisive, in which case the line is a substance too:

A line [is] a length that is breadthless.

El. 1, Def. 2

Euclid defines the circle by division as well, and hence as a substance:

A circle is a plane figure contained by one line [...] such that the straight lines falling upon it from one point among those lying within the figure [...] are equal to one another.

El. 1, Def. 15i

Since Euclid's definition of the circle is much more informative than most other genus-differentia definitions, Detel (1993, i, 216–218) takes it to be non-divisive. However, nothing prevents a definition by division from containing plenty of information. Euclid's definition of the circle is in fact a divisive definition with more than one differentia. While 'figure' is a genus of the circle, the remaining terms express differentiae. The first differentia—'plane'—restricts the circle to plane figures, and the second—'contained by one line'—to round plane figures. Only the third and last differentia—'such that the straight lines falling upon it from one point among those lying within the figure are equal to one another'—defines the circle itself. The other two differentiae narrow down the genus-term 'figure'. Consequently, the proximate genus of the circle is not simply 'figure', but 'plane figure contained by one line'.²³

The genera stated in Euclid's definitions by division are substances too, given that he defines them by division. For instance, while Euclid renders 'figure' as a genus of mathematical substance-kinds such as the circle (*El. 1, Def. 15i*), the square (*El. 1, Def. 22i*), the sphere (*El. x1, Def. 14*), and the cube (*El. x1, Def. 25*), he defines also the figure itself by division, and hence as a substance (by MARK S_{Div} , *El. 1, Def. 14*).

However, Euclid defines differentiae other than by division, and hence as non-substances (see Section 3.2 below). For instance, the attribute 'plane' is a differentia, as we saw, and it is defined other than by division (*El. 1, Def. 7*).

Why does Euclid define only substances such as the number, the line, and the circle by division (that is, through genus and differentia), but all non-substantial attributes of substances other than by division? Euclid's practice of

23 This shows that Euclid sometimes renders the genus by a complex expression, rather than by a noun alone.

definition appears to echo the Aristotelian view that only substances possess an essence (*ti ēn einai*), whereas non-substances do not possess an essence or possess an essence only derivatively.²⁴ Aristotle thus reserves the term 'substance' (*ousia*) for those items that can be subjects of essential predications. According to Aristotle, the subject of an essential predication is primarily a species or form (*eidos*).²⁵ A definition (*horos*, *horismos*) is an account (*logos*) that displays the essence of the definiendum,²⁶ and the definiens of a species consists only of its genus and differentia(e).²⁷

Accordingly, Euclid appears to take the essence of a substance to be stated primarily by the genus, given that the chief difference between Euclid's definitions of substances and those of non-substances lies in the fact that the former state a genus-predicate of the definiendum, whereas the latter do not.²⁸

Moreover, Aristotle takes the essence of a substance to be a cause (*aition*) of its demonstrable non-substantial attributes.²⁹ For this reason, Aristotle conceives of definitions as explanatory principles of scientific demonstrations.³⁰ Accordingly, Euclid appeals to definitions as deductive premises of his mathematical proofs.³¹ Euclid's distinction of non-substantial from substantial attributes in fact serves to distinguish demonstrable from indemonstrable attributes.

3.2 *Euclid's Negative Marker of Non-Substances: Non-Divisive Definition*

While Euclid defines only substances by division, he defines all non-substances other than by division. Non-divisive definition can therefore preliminarily serve as a negative marker of Euclid's definitions of non-substances:

24 *Metaph.* 1030a29–32; 1030b5–7; 1031a1–11.

25 *Metaph.* 1017b23–26.

26 *APo* 91a1; 93a29; *Metaph.* 1030a6–7; 27; 1031a11–12; 1042a17–21.

27 *APo* 83a39–b1; *Metaph.* 1037b30–32.

28 Aristotle, however, does not think that only substances can be subjects of genus-predications: for instance, Aristotle takes 'white' to be a subject of the genus 'colour' (*Categories* [= *Cat.*] 14a20–22), and so he frequently defines also non-substances by division. At any rate, Aristotle thinks that if a non-substance falls under a genus, then that genus must itself be a non-substance (for instance, 'colour' is a non-substantial genus), whereas the genus of a substance must itself be a substance.

29 *Metaph.* 1017b14–16. Aristotle's standard example is the essence of triangle being a cause of the fact that all and only triangles have an interior angle sum equal to two right angles (see, e.g., *Metaph.* 1025a30–32; cf. *El.* I, 32).

30 *APo* 72a14–24; 72b23–25; 89a16–19; 90b24–27; 96b21–25; 99a21–23.

31 For an overview of Euclid's appeals to definitions as deductive premises, see Acerbi 2007, 2543–2562.

MARK non- $S_{\text{non-Div}}$: If A is defined other than by division, then A is a non-substance.³²

For instance, 'straight' (for a line, *El.* 1, Def. 4) and 'plane' (for a surface, *El.* 1, Def. 7) are non-substances by MARK non- $S_{\text{non-Div}}$, and so are 'even' and 'odd' (for a number, *El.* VII, Def. 6–7), given that none of them is defined by division. In particular, Euclid's definition of a straight line does not state a genus-predicate of the definiendum, but only a differentia:

A straight line is that which lies equally with the points on itself.

El. 1, Def. 4

Euclid's definition of an even number too states only a differentia, but nothing that could be predicated as a genus of the definiendum:³³

An even number is that which is divisible in half.

El. VII, Def. 6

One might find it natural to regard definitions such as these as definitions by division, rather than as non-divisive definitions. Neuwirth (2015, 28), for instance, takes the definiendum in Euclid's definition of an even number to be the compound term 'even number', while he takes 'number' to be its genus and 'divisible in half' to be its differentia. Netz (1999, 91) takes the compound term 'straight line' to express the definiendum of Euclid's definition of a straight line, while Timpanaro-Cardini (1975, 183) takes 'line' to be its genus and 'lies equally with the points on itself' to be its differentia.

Against this, I suggest that the definiendum in Euclid's definitions of non-substances such as 'straight' and 'even' is in fact not expressed by the adjective-noun compound, but only by the adjective. My view is that, while it is true that Euclid's definitions of non-substances state a differentia of the definiendum,

32 Strictly speaking, MARK non- $S_{\text{non-Div}}$ is true only for the most part, given that there is one type of substance which Euclid defines other than by division (see in Section 4.6 below): while it is true that Euclid defines all non-substances non-divisively, he does not however define only non-substances non-divisively. Still, I argue in Section 3.4 below that Euclid has a positive marker of non-substances.

33 Although the Greek present passive participle *dihairoumenos* means 'divided', Riedlberger (2013, 229–230) and Acerbi (2017, 182) take it to have a potential sense in ancient mathematical contexts. Accordingly, it is usually translated by the modal form 'divisible' or 'can be divided' (see, e.g., Heath 1925, ii, 277; Vitrac 1994, 254).

they do not also state a genus. In Section 3.4 below, I argue that the logical function of the terms ‘line’ and ‘number’ in these definitions is not that of a predicate, but that of a subject term.

3.3 Exceptions

The *Elements* contains in total 27 putative counterexamples to MARK S_{Div} and to MARK non- $S_{non-Div}$: Euclid fails to state a genus in 8 of his divisive definitions, and he defines 19 substances other than by division. In what follows, however, I show that each of these exceptions can be explained.

The first kind of exception are Euclid’s divisive definitions of a substance that do not state a genus. The *Elements* features 8 divisive definitions of a substance that do not state a genus: those of the point, the surface, the boundary, the figure, the angle of a segment of a circle, the angle in a segment of a circle, the unit, and the solid (*El.* I, Def. 1; I, Def. 5; I, Def. 13–14; III, Def. 7–8i; VII, Def. 1; XI, Def. 1). However, I argue in Section 4.1 below that the absence of a concrete genus-term in Euclid’s definitions is not a counterexample to MARK S_{Div} , but Euclid’s distinctive marker of generic substances.

The second kind of exception are Euclid’s seemingly non-divisive definitions of a substance. Heiberg’s edition of the *Elements* contains 19 non-divisive definitions of mathematical objects that should be regarded as Euclidean substances, given that Euclid elsewhere defines the same type of object by division. To begin with, the circumference of the circle is a substance, given that it is a line, and lines are substances (by MARK S_{Div} , *El.* I, Def. 2). Still, the definition of ‘circumference’, which is inserted into Euclid’s definition of the circle (*El.* I, Def. 15i), is not divisive:

A circle is a plane figure contained by one line (which is called a circumference) [...].

El. I, Def. 15ii

One might construe *El.* I, Def. 15ii as a divisive definition by taking ‘circumference’ to be defined as “the line containing the circle”. However, it would be circular first to define ‘circle’ through ‘circumference’, and then to define ‘circumference’ through ‘circle’. Anyway, Def. 15ii is most likely inauthentic: while this definition is transmitted by the manuscripts, earlier ancient sources, such as Proclus’ commentary on Book I of the *Elements*, do not record it.³⁴

Moreover, the centre of the circle, of the semicircle, and of the sphere are substances, given that they are points, and points are substances (by

34 Cf. Heiberg 1903, 47; Heath 1925, i, 50; 184.

MARK S_{Div} , *El.* I, Def. 1). Still, Euclid does not define them by division (*El.* I, Def. 16; I, Def. 18ii; XI, Def. 16). However, I argue in Section 4.5 below that the distinctive way in which Euclid defines centre-points marks his demarcation of indivisible from divisible specific substances.

Also, the gnomon appears to be a Euclidean substance, given that gnomons are well-defined figures. Nonetheless, it is unclear what Euclid takes the genus and differentia of the gnomon to be:

Of every parallelogrammic area, let any of the parallelograms around its diameter together with the two complements be called a gnomon.

El. II, Def. 2

Pseudo-Hero of Alexandria's *Definitions* (= *Deff.*) features a divisive definition of 'gnomon' as "every [figure] which, when added to any [figured] number or figure whatsoever, makes the whole similar to that to which it is added" (*Deff.* 58, 44, 13–14). This provides a clue as to how to analyze *El.* II, Def. 2 in divisive fashion. Euclid's definition is narrower than Pseudo-Hero's, as Euclid restricts gnomons to parallelogrammic areas. While the genus of Pseudo-Hero's definition appears to be 'figure' in general, 'parallelogram' figures as the genus in Euclid's definition of 'gnomon'. Its differentia will, then, be: 'placed around the diameter of a parallelogrammic area together with the two complements'. Therefore, *El.* II, Def. 2 can be understood as a definition by division, after all.

Furthermore, Euclid's first definition of 'ratio ex aequali' (*El.* v, Def. 17i) hardly qualifies as a definition by division either. However, Euclid's second, alternative definition of 'ratio ex aequali' is divisive, taking the term 'taking' (*lēpsis*) to be its genus, in which case ratios ex aequali are Euclidean substances (by MARK S_{Div} , *El.* v, Def. 17ii). This is also the way in which Euclid defines the other transformations of a ratio (*El.* v, Def. 12–16). Since Def. 17i is the only non-divisive definition among these, it can simply be dismissed as an interpolation.³⁵

The remaining 13 exceptions are Euclid's definitions of the medial line (*El.* x, 21) and of the twelve subspecies of irrational lines (*El.* x, 36–41; 73–78). Euclid defines the irrational lines other than by division, although lines are substances (by MARK S_{Div} , *El.* I, Def. 2). Given that I focus in this paper on those of the *Elements'* definitions which are expressly labelled as "Definitions" (*Horoi*) and are listed before the proof-propositions, I would like to set this group of definitions aside, as they are stated as the conclusions of proof-propositions.

35 Cf. Vitrac 1994, 54.

3.4 *Euclid's Positive Marker of Non-Substances: Definition by Addition*

Euclid does not define non-substances by stating a genus of the definiendum, but by a certain addition. That which is added to the definition is a reference to the name or account of that substance of which the non-substantial definiendum is an attribute. Euclid defines all and only non-substances by such an addition. Definition by addition is in fact Euclid's positive marker of non-substances:

MARK non-S_{Add}: If *A* is defined by addition of the primary subject *B*,
then *A* is a non-substance.

For instance, non-substantial attributes such as 'straight' (for a line, *El.* VI, Def. 4), 'plane' (for a surface, *El.* I, Def. 7), 'part of a magnitude' (for a magnitude, *El.* V, Def. 1), 'part of a number' (for a number, *El.* VII, Def. 3), and 'even' and 'odd' (for a number, *El.* VII, Def. 6–7) are not only defined other than by division, but also by addition of the primary subject of the respective definiendum. In Euclid's definition of 'even', the primary subject is the term 'number':

Artios arithmos estin ho dicha dihairoumenos.

An even number is that which is divisible in half.

El. VII, Def. 6

The definiendum of Euclid's definition of 'even' is expressed by the adjective "even" (*artios*), while the complex verbal phrase "divisible in half" (*dicha dihairoumenos*) expresses the differentia of 'even'. But what tells us that the noun "number" (*arithmos*) expresses the primary subject of 'even', rather than its genus? Let us compare Euclid's definition of 'even' to his divisive definition of 'number':

Arithmos de to ek monadōn sugkeimenon plēthos.

A number [is] that plurality which is composed of units.

El. VII, Def. 2

One might be led to think that both of these accounts are equally definitions by division, given that not only the definiens of Euclid's definition of 'number', but also that of his definition of 'even', contains a noun in the nominative case that might be taken to be the genus of the respective definiendum: "plurality" (*plēthos*) in the definition of 'number', "number" (*arithmos*) in that of 'even'. The term 'plurality' is clearly rendered as a genus in Euclid's definition

of 'number'. But can Euclid equally be regarded to render the term 'number' as a genus of 'even'?

In Euclid's definition of 'number', the term 'plurality' is contained as a genus-predicate in the definiens; but the term 'plurality' does not also occur in the definiendum. In his definition of 'even', by contrast, the term 'number' occurs twice: it is expressly stated alongside the definiendum 'even', and it is implicitly repeated in the definiens. The Greek masculine article *ho* ("the", here translated as "that which") elliptically indicates that the noun "number" (*arithmos*) is also present in the definiens.³⁶

The fact that 'number' occurs on both sides of Euclid's definition of 'even' indicates that the term 'number' is not contained as a genus-predicate in the definiens of 'even'. Rather, the reference to the term 'number' in Euclid's definition of 'even' fixes the domain for which that attribute is defined: magnitudes too are divisible in half, but 'even' is defined only for numbers, not for magnitudes.

While 'plurality' is predicated as a genus of 'number', 'number' is not predicated as a genus of 'even'. The term 'number' is in fact not predicated of 'even' at all.³⁷ Instead, 'even' is predicated of 'number'. In particular, 'even' is predicated non-universally, but exclusively of 'number', since while not every number is even, only numbers can be even. Therefore, 'number' is not a genus-predicate in Euclid's definition of 'even', but the primary subject of 'even'.

More generally, definitions by addition do not refer to a genus-predicate of the non-substantial definiendum, but to a primary subject of which the definiendum is predicated. Both the genus and the primary subject are substances, but the primary subject has an entirely different logical function than the genus: while the genus is not a subject, but only a predicate of the substantial definiendum, the primary subject is not a predicate, but only a subject of the non-substantial definiendum. In Euclid's definitions by division, it is in the definiens, and only in the definiens, that Euclid refers to a genus as a predicate of the substantial definiendum. In Euclid's definitions by addition, however, the primary subject occurs in both the definiendum and the definiens.

36 Whenever Euclid renders definitions by addition in conditional form, the repetition of the primary subject term in the definiens is elliptically indicated by a pronoun (e.g., *El.* IV, Def. 7) or the verb form (e.g., *El.* V, Def. 1–2; VII, Def. 3–5).

37 While 'number' is not predicated of 'even', 'number' is predicated as a genus of numerical species (that is, specific numbers) such as two and three. Numerical species, however, do not belong to the subject-matter of Euclid's arithmetic, which is restricted to numerical attributes.

Therefore, in addition to defining non-substances other than by division, Euclid distinctively defines non-substances by addition of their primary subject.

Euclid's reason for defining all and only non-substances by addition, instead of through a genus, can plausibly be regarded to lie in the Aristotelian view that non-substances depend upon substances. While Aristotle excludes non-substances from possessing an essence, he admits that non-substances are still definable in some other way, namely, by an addition (*prosthesis*): definitions of non-substances refer to that substance of which they are non-substantial attributes, that is, their primary subject.³⁸ Aristotle takes all items in non-substantial categories (qualities, quantities, relatives, etc.) to be defined by such an addition.³⁹ Euclid's practice of defining all non-substances by addition converges with Aristotle's theory of definition also insofar as Aristotle takes the primary subject not to be contained as a predicate in the definiens,⁴⁰ but to be present on both sides of the definition.⁴¹

3.5 *Euclid's Second Positive Marker of Substances: The Primary Subject*

While having a primary subject is indicative of non-substances, being the primary subject is indicative of substances. Since the primary subject in Euclid's additive definitions of non-substances is always a substance, definition by addition yields Euclid's second positive marker of substances:

MARK S_{PS}: If *B* is defined by addition of the primary subject *A*, then
 A is a substance.

For instance, 'line' is the primary subject of non-substantial attributes such as 'straight' (*El.* I, Def. 4), and 'surface' is the primary subject of 'plane' (*El.* I, Def. 7). Therefore, both the line and the surface are Euclidean substances by MARK S_{PS}. Equally, the number is a Euclidean substance by MARK S_{PS}, given that 'number' is the primary subject of non-substantial numerical attributes such as 'even', 'odd', 'prime', and 'composite' (*El.* VII, Def. 6–7; 12; 14).

This reflects the Aristotelian view that only substances can be the ultimate subjects of non-substantial attributes. According to Aristotle's *Categories*,

38 *Metaph.* 1028a35–36; 1030b23–24.

39 *Metaph.* 1031a2–4.

40 *Metaph.* 1029b29–33.

41 *Metaph.* 1030b30–34; 1031a4–5; cf. *Topics* (= *Top.*) 149a29–37.

substances are the primary subjects of inherence: non-substances inhere in substances, whereas substances do not inhere in non-substances.⁴²

3.6 Exceptions

The *Elements* seems to contain two kinds of exception to MARK S_{PS} and MARK non- S_{Add} . The first kind of exception are those of Euclid's definitions by addition which seem to have a non-substance as their primary subject term. For instance, 'straight' is a non-substance by MARK non- $S_{non-Div}$, as Euclid's definition of a straight line is non-divisive (*El.* I, Def. 4). By contrast, 'straight' might seem to be a substance by MARK S_{PS} , given that 'straight lines' serves as the primary subject of non-substantial attributes such as 'parallel' (*El.* I, Def. 23), 'to touch a circle' (*El.* III, Def. 2; this is in fact Euclid's definition of a tangent), 'to be fitted into a circle' (*El.* IV, Def. 7), 'to have been cut in extreme and mean ratio' (*El.* VI, Def. 3), 'commensurable in square' (*El.* X, Def. 2i), and 'at right angles to a plane' (*El.* XI, Def. 3). The fact that 'straight lines' is the primary subject of a non-substance seems to violate MARK S_{PS} , according to which only substances can be primary subjects of non-substances.

However, the fact that compounds of a substance and a non-substance—such as 'straight line'—can serve as the primary subject of another non-substance—such as 'parallel'—does not imply that such compounds constitute certain substance-kinds. Straight lines are substances, not in virtue of being straight, but in virtue of being lines. Thus, 'parallel' is ultimately an attribute of 'line', not of 'straight'. Parallel straight lines are substances because parallel lines are straight lines, straight lines are lines, and lines are substances (by MARK S_{Div} , *El.* I, Def. 2).

The second kind of exception to MARK S_{PS} and MARK non- S_{Add} are Euclid's divisive definitions of substances that seem to look like additive definitions. The *Elements* records 20 such cases, namely, Euclid's definitions of the kinds of angle (*El.* I, Def. 8–10i; 11–12; XI, Def. 11i–ii), of the kinds of transformation of a ratio (*El.* V, Def. 12–17), of the kinds of trilateral figure (*El.* I, Def. 20i–21iii), and of the kinds of cone (*El.* XI, Def. 18ii–iv). The definiendum of each of these is expressed by an adjective plus a noun in the nominative case. This is exactly how Euclid usually defines non-substances. Still, each of these definitions also states a genus-predicate, which is how Euclid normally defines a substance. For instance, Euclid seems to define the equilateral triangle both by division and by addition:

42 Cat. 3a7–20.

Tōn de tripleurōn schēmatōn isopleuron men trigōnon estī to tas treis isas echon pleuras

Of trilateral figures, an equilateral triangle is that which has the three sides equal.

El. I, Def. 20i

While ‘equilateral triangle’ is the definiendum, ‘trilateral figure’ (defined in *El. I, Def. 19ii*) is the genus-predicate of ‘equilateral triangle’. In this respect, *Def. 20i* is a divisive definition of a substance. Yet, the term ‘triangle’ is simultaneously stated as a subject of ‘equilateral’, whereby a reference to a subject term of the definiendum is characteristic of Euclid’s additive definitions of non-substances.

Definitions such as these seem to blend the two mutually exclusive registers of definition by division and definition by addition with one another. However, the impression of additive definition arises only because Euclid here fails to introduce a name for the definiendum. Instead of calling the definiendum of *Def. 20i* by one special name, Euclid uses the phrase ‘equilateral triangle’, just as he calls the definiendum of *Def. 19ii* “trilateral figure”, rather than “triangle”. While it is true that ‘triangle’ is a subject of ‘equilateral’, it is not its primary subject, given that the term ‘triangle’ is not present on both sides of this definition, but occurs only in the definiendum. Also, the genus ‘trilateral figure’ is stated only in the definiens. Therefore, *Def. 20i* is not a definition by addition, but a definition by division only. This applies equally to the other 19 cases.

The definitions in which Euclid fails to introduce a special name for the definiendum can be taken to reflect ancient controversies about the ontological status of ratios, angles, and figures such as triangles.

Take, for instance, the question of what category trilateral figures belong to. Aristotle treats the triangle as a primary subject of non-substantial attributes.⁴³ This suggests that Aristotle regards triangles as (quasi-)substances, a position which is shared by Euclid, who defines ‘trilateral figure’ and each of the different kinds of triangle by division (by MARK S_{DIV}, *El. I, Def. 19ii*; *Def. 20i–20iii*). Still, Aristotle takes triangles to be qualities,⁴⁴ and hence non-substances. In late antiquity (long after Euclid), Porphyry contests Aristotle’s view that triangles

43 *APo* 90a9–14; *Metaph.* 1025a30–b2.

44 *Cat.* 10a14–15; 11a5–14; *APo* 76a31–36; cf. *Metaph.* 1020a33–b1; 1024a36–b4. Aristotle takes also ‘figure’ (*schēma*) to be a quality: *Cat.* 10a11–16.

are qualities,⁴⁵ while Plotinus,⁴⁶ Simplicius,⁴⁷ and Proclus⁴⁸ respectively argue that triangles are qualities in one respect, but quantities in another.

Also, the ontological status of angles was controversially debated among ancient philosophers.⁴⁹ While Aristotle⁵⁰ and Eudemus⁵¹ treat angles as qualities, Syrianus and Proclus treat angles as qualities, quantities, and relatives at once.⁵² Proclus argues that Euclid takes angles to be relatives because Euclid defines angles as inclinations,⁵³ whereas Heath (1925, i, 178) and Artmann (1999, 6) argue that Euclid treats angles as magnitudes, and hence as quantities. Euclid, however, defines angles and inclinations by division, and hence as Euclidean substances (by MARK S_{Div}, *El.* I, Def. 8–10i; Def. 11–12; XI, Def. 5–6; Def. 11i–ii).

Moreover, Aristotle reports a controversy concerning the question of whether or not ratios are substances.⁵⁴ Aristotle himself appears to take ratios to belong to the category of relative (*pros ti*),⁵⁵ and hence does not regard them as substances. However, while Euclid takes ratios to be some kind of relation (*poia schesis*), he defines 'ratio' by division, and hence as a Euclidean substance (by MARK S_{Div}, *El.* v, Def. 3).

4 Euclid's Distinction between Different Kinds of Substance

Let me now turn to my second main claim, which concerns Euclid's diverse subdivisions of substances. Specifically, I argue that Euclid distinguishes between specific and generic substances, as well as between indivisible and divisible specific substances. This is indicated by the fact that Euclid systematically defines each of these different kinds of mathematical substance in a distinctive way. Euclid defines both specific and generic substances by division, but he defines specific substances with a concrete genus-term, whereas he defines generic substances without a concrete genus-term. Indivisible specific substances, however, are defined in an entirely different manner.

45 Porphyry, *On Aristotle Categories* 132,20–133,6.

46 Plotinus, *Enneads* VI.3, 14,7–36.

47 Simplicius, *On Aristotle Categories* 153,3–5.

48 *in Eucl.* 123,24–124,2.

49 Cf. Heath 1925, i, 176–178.

50 *Cat.* 10a10–24; *Metaph.* 1020a35–b8; *Phys.* 188a25–26.

51 Cf. Proclus, *in Eucl.* 125,4–13.

52 *in Eucl.* 123,19–125,3.

53 *in Eucl.* 125,13–14.

54 *Metaph.* 1001b26–28.

55 Cf. Szabó 1969, 105.

4.1 *Euclid's Distinction between Generic and Specific Substances*

As we already saw in Section 3.3 above, Euclid's *Elements* features eight divisive definitions in which there is no genus stated. However, I take these exceptions not to be counterexamples to MARK S_{Div} , but to introduce a subdivision of different kinds of substance. I call 'generic' those substances that do not have any specific features, and 'specific' those substances that are certain kinds of generic substances. For instance, the circle and the sphere are specific kinds of figure, respectively, whereas the figure itself is not.

68 of the 76 divisive definitions stated in the *Elements* feature a concrete genus-term, whereas 8 of them do not. In turn, 6 of these latter 8 divisive definitions without a genus define generic items, whereas the remaining 2 define non-generic, specific items. (I discuss each of these exceptions below.)

Specific substances are thus indicated by the presence of the genus, whereas the absence of the genus within the framework of a definition by division is indicative of generic substances.

4.2 *Euclid's Marker of Generic Substances: Definition by Division without a Concrete Genus*

The fact that Euclid regularly defines generic substances by division without stating their genus yields Euclid's marker of generic substances:

MARK $GS_{Div-non-Gen}$: If A is defined by division, but the genus of A is omitted, then A is a generic substance.

By MARK $GS_{Div-non-Gen}$, the point (*sēmeion*, *El.* I, Def. 1), the surface (*epipha-neia*, *El.* I, Def. 5), the boundary (*horos*, *El.* I, Def. 13), the figure (*schēma*, *El.* I, Def. 14), the solid (*stereon*, *El.* XI, Def. 1), and the unit (*monas*, *El.* VII, Def. 1) are generic substances. This is indicated by the fact that each of these mathematical objects is defined by division, but without stating a genus. The presence of an article without a corresponding noun suggests that these definitions are indeed supposed to be divisive. Instead of providing a concrete genus-expression, Euclid only states "that" as the genus of generic substances. He thereby reserves an empty slot for the unnamed genus. For instance, in Euclid's definition of the generic point, the relative article indicates that this account is meant to be a definition by division, although it does not state a concrete genus-term:

Sēmeion estin, hou meros outhen.

A point is that which has no part.

El. I, Def. 1

The article without a corresponding noun also serves as a place marker for the genus in Euclid's definition of the generic solid:⁵⁶

Stereon esti to mēkos kai platos kai bathos echon.

A solid is that which has a length and a breadth and a depth.

El. XI, Def. 1

Euclid's definitions of generic substances differ from ordinary definitions by division in that they do not state a concrete genus-predicate. The absence of the genus is something they have in common with Euclid's definitions of non-substances. Still, Euclid's definitions of generic substances differ from his definitions of non-substances in that his definitions of generic substances do not state a primary subject of the definiendum. Rather, they are definitions by division the genus of which is omitted. More precisely, their genus-expression is not a concrete term, but an empty slot.

Why does Euclid not state a concrete genus in the divisive definitions of generic mathematical items? Does Euclid take them not to fall under a genus? Does he take them to be highest genera such that there is no genus that can be predicated of them? The fact that Euclid defines generic substances by division suggests that he presumes generic substances to be in a certain genus, even if he does not mention their genera.⁵⁷

Euclid's reason for omitting the genera of generic substances can plausibly be regarded to lie in the view that these genera fall outside the scope of the respective mathematical discipline. The most generic items of a particular scientific discipline constitute the subject-genus of that discipline, whereas the genera of the generic substances extend beyond that subject-matter. Euclid avoids mentioning these genera in order not to cross the boundaries of mathematics.

Aristotle, by contrast, usually states concrete terms as the genera of generic mathematical substances. Still, Aristotle does not state these concrete genera

56 Euclid's definition of the solid is supplemented by the additional statement "And a limit of a solid is a surface" (*El. XI, Def. 2*). This does not supply a genus either, but qualifies the differentia: a solid is not simply something three-dimensional, but it is necessarily limited by a surface or surfaces; otherwise it would be nothing but unlimited extension.

57 Determining the genus of a generic mathematical object is a rather difficult task. Majer (1881, 21–22) suggests 'space' ("Räumlichkeit im allgemeinen", "Örtlichkeit") to be the absent genus in Euclid's definition of the point, and of geometrical objects in general. The Greek term for space is *topos* (often translated as 'place' or 'location'). Euclid uses the word *topos* only twice in the *Elements* (in *El. III, 16*), in the sense of 'the space between two lines'. However, since neither points nor magnitudes are spaces between different magnitudes, *topos* does not seem to be the genus of Euclid's generic geometrical items.

in mathematical, but only in philosophical contexts. As the genus of both ‘one’ (*hen*) and ‘point’ (*stigmē*), for instance, he mentions ‘substance’ (*ousia*),⁵⁸ as well as ‘unit’ (*monas*).⁵⁹ Aristotle also, however, mentions the term ‘point’ as the genus of ‘unit’.⁶⁰ Like Euclid (*El.* I, Def. 2), Aristotle takes ‘length’ (*mēkos*) to be the genus of ‘line’, but Aristotle also states ‘breadth’ (*platos*) as the genus of ‘surface’ (*epiphaneia*), and ‘depth’ (*bathos*) as the genus of ‘body’;⁶¹ whereas Euclid takes these terms to be part of the respective differentiae of ‘surface’ (*El.* I, Def. 5) and ‘solid’ (*El.* XI, Def. 1). Moreover, Aristotle takes ‘magnitude’ (*megethos*) to be the genus of ‘figure’ (*schēma*).⁶²

Furthermore, Aristotle does not take articles without a corresponding noun to be sufficiently indicative of an omitted genus, or that Aristotle takes all divisive definitions to require a concrete genus-term. This is suggested by the fact that Aristotle asserts that the putative Platonic definition of ‘body’ (*sōma*) as “that which has three dimensions” (*to echon treis diastaseis*) fails to state a genus.⁶³

4.3 *Euclid's Marker of Specific Substances: Definition by Division with a Concrete Genus*

Since MARK_{GS_{Div-non-Gen}} singles out the definitions of generic substances among Euclid's definitions of substances in general, all remaining divisive definitions are those stating a concrete genus. This yields Euclid's marker of non-generic, specific substances:

MARK_{SS_{Div-Gen}}: If *A* is defined by division, and the genus of *A* is expressed by a concrete term, then *A* is a specific substance.

For instance, the circle and the square are specific substances by MARK_{SS_{Div-Gen}}, as Euclid states their respective genus. The genus of ‘circle’ is ‘plane figure contained by one line’ (*El.* I, Def. 15i), and the genus of ‘square’ is ‘quadrilateral figure’ (*El.* I, Def. 22i). Moreover, the genus of ‘quadrilateral figure’ is ‘rectilinear figure’ (*El.* I, Def. 19iii), the genus of which is in turn simply ‘figure’ (*El.* I, Def. 19i). The figure, however, is a generic substance (by MARK_{GS_{Div-non-Gen}}, *El.* I, Def. 14), and not further generalized.

⁵⁸ *APo* 87a36.

⁵⁹ *An.* 409a6. Proclus (*in Eucl.* 95,21) ascribes this view to the Pythagoreans.

⁶⁰ *Metaph.* 1084b26–27.

⁶¹ *Metaph.* 1020a13–14.

⁶² *An.* 425a18.

⁶³ *Top.* 142b22–29.

Euclid's distinction of specific substances from generic ones thus serves to distinguish those substances which can be further defined in terms of higher genera from those which cannot (within mathematics, at least). While generic substances demarcate the subject-genus of a mathematical discipline and provide genera for the definitions of specific substances, specific substances are Euclid's actual substance-kinds (species).

4.4 *Exceptions*

The *Elements* features two kinds of exception to MARK_{GS_{Div-non-Gen}} and MARK_{SS_{Div-Gen}}. The first kind of exception are Euclid's divisive definitions of specific substances that fail to state a concrete genus-term. The *Elements* contains two such definitions: those of 'angle of a segment of a circle' and of 'angle in a segment of a circle' (*El.* III, Def. 7–8i). They fail to state a concrete genus-term, not because angles in circles are among the most generic items in geometry, but rather because it is notoriously difficult to say what angles are. According to Euclid's definitions of the plane angle (*El.* I, Def. 8) and the solid angle (*El.* XI, Def. 11i), the genus of 'angle' is 'inclination'.⁶⁴ However, while Euclid defines angles in terms of 'inclination' (*klisis*), he circularly defines inclinations in terms of 'angle' (*gōnia*, *El.* XI, Def. 5–6). Thus, Euclid's failure to state a genus-term in *El.* III, Def. 7–8i is not indicative of his definitions of generic substances, but of a generic uncertainty concerning the ontology of angles (see also Section 3.6 above).

The second kind of exception are Euclid's divisive definitions of a generic substance that state a concrete genus. There are two definienda in Euclid's *Elements* which one would expect to be included in Euclid's list of generic substances, but which are nonetheless defined through a concrete genus: these are 'line' and 'number.' Euclid renders 'length' (*mēkos*) as the genus of 'line' (*El.* I, Def. 2), although he neither states a concrete genus of 'point' nor of 'surface' nor of 'solid'. Equally, although Euclid states no concrete genus of 'unit', he renders 'plurality' (*plēthos*) as the genus of 'number' (*El.* VII, Def. 2).

So, why does Euclid state a concrete genus of 'line' and 'number'? The line and the number share one significant feature with one another which they do not share with any of the other generic substances. The line is the most fundamental divisible generic substance in geometry, while the number is the most fundamental divisible generic substance in arithmetic. Just as Euclid's definition of the line immediately follows his definition of the only indivisible

64 While Euclid defines 'inclination' by division, and hence as a substance (by MARK_{S_{Div}}, *El.* XI, Def. 5–6), Aristotle (*APo* 76b9–12) takes 'inclination' (*to keklasthai*) to be an intrinsic accident of certain substances.

item in geometry: the point, so too Euclid's definition of the number immediately follows his definition of the only indivisible item in arithmetic: the unit. Consequently, the exceptional status of the line and the number among generic substances is reflected by the exceptional formulations of their respective definitions.

4.5 *Euclid's Distinction between Divisible and Indivisible Specific Substances*

Euclid provides a subdivision of specific substances into divisible and indivisible ones. Here I speak of 'indivisible', not in the sense of 'individual', but in the quasi-physical sense of 'having no parts'. Indivisible specific substances are those without parts, while those with parts are divisible. Since Euclid's definitions of divisible specific substances coincide with those divisive definitions which state a concrete genus, his marker of specific substances (MARK SS_{Div-Gen}) is in fact a marker of divisible specific substances only. Euclid's definitions of indivisible specific substances, however, are defined in a distinctive manner.

As I have already mentioned in Section 3.3 above, Euclid defines the only three specific centre-points of certain round figures considered in the *Elements* other than by division. This is surprising, given that he defines the generic point by division (*El.* 1, Def. 1). Euclid distinguishes centre-points not only from the generic point, but also from all of the other specific geometrical substances defined in the *Elements*. Indeed, Euclid's definition of the generic point as "that which has no parts" (*El.* 1, Def. 1) implies that points are the only indivisible items in geometry. While Euclid does not define centre-points by division, he does not define them by addition either. Instead, he defines them by identification. By 'identification' I mean in this context that the definiens of one definition is expressly identified with the definiendum of another definition.

4.6 *Euclid's Marker of Indivisible Specific Substances: Definition by Identification*

The fact that Euclid defines all and only specific points by identification yields Euclid's marker of indivisible specific substances:

MARK ISS_{Id}: If *A* is defined by identification, then *A* is an indivisible specific substance.

Specifically, Euclid initially identifies the single point (*hen sēmeion*) referred to in the definiens of his definition of the circle (*El.* 1, Def. 15i) with the definiendum of *El.* 1, Def. 16, saying that this point is called 'a centre of the circle':

And the point [*scil.* referred to in *El.* 1, Def. 15i] is called a centre of the circle.

El. 1, Def. 16

Euclid's definitions of the other two centre-points refer to the centre of the circle too:

And a centre of the semicircle is the same as that of the circle.

El. 1, Def. 18ii

And a centre of the sphere is the same as that of the semicircle.

El. x1, Def. 16

Euclid identifies the definiens of *El.* 1, Def. 18ii with the definiendum of *El.* 1, Def. 16, before then also identifying the definiens of *El.* x1, Def. 16 with the definiendum of *El.* 1, Def. 18ii. Therefore, all three centre-points are identified with one another.

Euclid's lexical choice of the verb in *El.* 1, Def. 18ii and x1, Def. 16 supports this: by using a form of 'to be the same' (*einai to auto*) to connect the definiendum with the definiens, Euclid expressly states that the three centre-points stand to one another in the relationship of identity. *El.* 1, Def. 18ii and x1, Def. 16 are the only definitions in the *Elements* in which a form of 'to be the same' is used as the connecting verb.

Definition by identification makes a difference to both definition by division and definition by addition. Definitions by identification neither state a genus that is predicated of the definiendum nor a primary subject of which the definiendum is predicated. Instead, they simply identify their definiendum with the definiens of another, previously stated definition. One might want to object that definition by identification is not distinctive of Euclid's definitions of centres because (ideally) every definition is a certain identity statement. However, only Euclid's definitions of centre-points state an identity between the definiendum of one definition with the definiens of another: they are not only intra-, but also interdefinitional identity statements.

4.7 *Exceptions*

There are two putative counterexamples to MARK ISS_{Id}. Although the three centre-points are the only indivisible specific substances defined by Euclid, the text of the *Elements* contains two definitions by identification of which the definiendum is not an indivisible specific substance.

The first exception is the definition of the circumference of the circle (*El. I*, Def. 15ii), which is inserted into Euclid's definition of the circle (*El. I*, Def. 15i). Def. 15ii identifies its definiendum with the single line (*mia grammē*) referred to in Def. 15i, using a form of 'to call' (*kalein*) as the connecting verb. While the circumference is a specific substance, it is not indivisible, given that lines are divisible. Therefore, Def. 15ii is indeed an exception to MARK 1SS_{Id}. However, as I already pointed out in Section 3.3 above, Def. 15i can simply be dismissed as an interpolation.

The second exception, which is Euclid's definition of proportional magnitudes (*El. v*, Def. 6), is most likely authentic, though.⁶⁵ Euclid stipulates that those magnitudes which are in the same ratio to one another (*en tōi autōi logōi*, famously defined in *El. v*, Def. 5) are to be called "proportional" (*analogon*, *El. v*, Def. 6), again using a form of *kalein* as the connecting verb. Since Euclid identifies the definiens of Def. 6 with the definiendum of Def. 5, Def. 6 is indeed a definition by identification. Still, since Euclid defines 'to be in the same ratio' by addition of the primary subject 'magnitudes', proportionality is not an indivisible specific substance, but a non-substantial attribute of magnitudes (by MARK non-S_{Add}, *El. v*, Def. 5). However, this exception can be bypassed by the fact that Def. 6 does not define a new attribute in its own right, but simply introduces the shorter term 'proportional' for 'to be in the same ratio' defined in Def. 5. The purely abbreviatory function of Def. 6 is emphasized by the fact that, within this definition, 'proportional' is stated after the definiens, whereas Euclid usually states the definiendum before the definiens.

5 Euclid's Distinction between Different Kinds of Non-Substance

Let me finally turn to my third main claim. Besides drawing a distinction between substances and non-substances on the one hand and multiple subdivisions of substances on the other, Euclid also provides a subdivision of non-substances. Specifically, I argue that Euclid distinguishes between *differentiae* and *relatives*.

5.1 *Euclid's Distinction between Differentiae and Relatives*

Euclid defines *differentiae* in a way that is systematically different from the way in which he defines *relatives*. This shows that Euclid draws a sharp boundary between these two kinds of non-substance.

65 Heath 1925, ii, 129.

Following Aristotle's notion of 'differentia' (*diaphora*), I call 'differentia' that non-substantial attribute which is predicated exclusively of its subject,⁶⁶ and which is predicated essentially and universally of the species,⁶⁷ but non-universally of the genus.⁶⁸ For instance, since every triple is odd, 'odd' is predicated universally and essentially—just like the genus 'number'—of numerical species such as number three.⁶⁹ However, since not every number is odd, 'odd' is neither a genus of 'three', nor a species of 'number'. Instead, since only numbers are odd, 'odd' is a differentia resulting from the division of the genus 'number'.⁷⁰

Following Aristotle's notion of 'relative' (*pros ti*), I call 'relative' that kind of non-substantial attribute which existentially depends on its correlative opposite, such as the correlative terms 'double' and 'half'.⁷¹

Euclid defines differentiae by a preverbal position of the primary subject term, but relatives by a postverbal position of the primary subject term. By 'preverbal position' I mean that the noun expressing the primary subject of the non-substantial definiendum is put before the verb connecting the definiendum with its definiens. Accordingly, by 'postverbal position' I mean that the primary subject is put after the connecting verb. While Greek word order allows for the noun to occur in both positions, Euclid systematically alters the position of the noun to introduce a metaphysical difference. This difference in the mere word order is Euclid's characteristic way of distinguishing between different kinds of non-substance.

In terms of numbers, all of the *Elements*' 69 definitions by addition define non-substances, as we saw. 63 of these 69 definitions of non-substances have preverbal word order, while 6 have postverbal word order instead. The 63 definitions with preverbal word order coincide with Euclid's definitions of differentiae.⁷² (One differentia is defined with a postverbal position of the primary subject term, but I explain this irregularity in Section 5.4 below.) The remaining 5 additive definitions with postverbal word order coincide with Euclid's definitions of relatives. Thus, 64 of the 69 additive definitions are of

66 Cf. *Top.* 149a25–33.

67 While Aristotle appears to exclude differentiae from being essential predicates in the *Metaphysics* (1030a17–27; 1031a1–2), differentiae are assumed to be predicated essentially of the species in the *Topics* (103b14–16; 107b19–37; 139a24–35; 122b16–17; 128a23–26; 143b19–20; 144b16–24; 153a11–22; b14–18; 154a27–28).

68 Cf. *Top.* 144b26–30.

69 *APo* 96a27–96b5.

70 Cf. *Top.* 122b19; 123a11–14; 142b10.

71 *Cat.* 6a36–b6; 8a31–32. Cf. already Plato, *Sophist* 255c14.

72 This includes 22 definitions in which the definiendum is expressed by a complex verbal phrase.

differentiae, and 5 are of relatives. As a result, while Euclid uses definition by addition of the primary subject to single out non-substances in general, his subdivision of particular kinds of non-substance is indicated by the position of the primary subject term.

5.2 *Euclid's Marker of Differentiae: Preverbal Position of the Primary Subject Term*

The fact that Euclid defines all and only differentiae with a preverbal position of the primary subject term yields Euclid's marker of differentiae:

MARK Diff_{PS-pre}: If *A* is defined by addition of the primary subject *B*, and *B* has a preverbal position, then *A* is a differentia.

For instance, by MARK Diff_{PS-pre}, Euclid's definitions of 'straight' (for a line, *El.* I, Def. 4), 'plane' (for a surface, *El.* I, Def. 7), 'even', and 'odd' (for a number, *El.* VII, Def. 6–7) are definitions of differentiae. Let us take once more a close look at Euclid's definition of 'even':

Artios arithmos estin ho dicha dihairoumenos.

An even number is that which is divisible in half.

El. VII, Def. 6

The noun 'number' (*arithmos*) is here put before the verb form 'is' (*estin*). Since Euclid states the term 'number', which is the primary subject of 'even', before the connecting verb, 'even' is a differentia, by MARK Diff_{PS-pre}.

Like Euclid, Aristotle too appears to take differentiae to be non-substantial attributes of substances, given that Aristotle takes the differentia to state a certain quality (*poion ti*) of its primary subject,⁷³ and he also categorially distinguishes differentiae from relatives.

Still, Aristotle cannot be found to indicate this distinction by the position of the primary subject term. While Aristotle formulates some of his definitions with a preverbal position of a part of the definiens, these are not definitions by addition, but by division, and their preverbally positioned term is not a subject of the definiendum, but a genus-predicate.⁷⁴ In addition, Aristotle's formula-

73 *Cat.* 3a21–22; *Metaph.* 1020a33–b2; b14–17; 1024b–6; 1068b19; *Phys.* 226a28. (Yet, this is a controversial issue among scholars on Aristotle.)

74 See, e.g., Aristotle's definitions of 'harmony' (*An.* 407b32–33), 'point' (409a6), 'voice' (420b5–6), 'definition' (*Metaph.* 1038a29–30), 'courage' (*EN* 116a10–11), 'prudence' (117b24–25), 'proportion' (1130a31–32), 'justice' (1133b32–33), 'forgiveness' (1043a23–24), and 'perfect happiness' (1178b7–8).

tions of definitions display no uniform treatment of pre- and postverbal position of nouns.⁷⁵

5.3 *Euclid's Marker of Relatives: Postverbal Position of the Primary Subject Term*

The fact that Euclid defines all and only relatives by a postverbal position of the primary subject term yields Euclid's marker of relatives:

MARK Rel_{PS-post}: If *A* is defined by addition of the primary subject *B*,
and *B* has a postverbal position, then *A* is a relative.

Euclid's definitions of relatives are those of 'part of a magnitude' and 'multiple of a magnitude' (for a magnitude, *El.* v, Def. 1–2), as well as of 'part of a number', 'parts of a number', and 'multiple of a number' (for a number, *El.* vii, Def. 3–5). Each of these involves a pair of correlative opposites: the part and the whole, the less and the greater, the measure and the multiple. The primary subject term in Euclid's definitions of relatives does not have a preverbal, but a postverbal position. (In addition, Euclid defines relatives by a partial postverbal position of the definiendum: while part of the definiendum is put before the connecting verb, another part is put after both the connecting verb and the primary subject term.) Let us take a closer look at Euclid's definition of 'part of a number':

Meros estin arithmos arithmou ho elassōn tou meizonos, hotan katametretōn meizona.

A number is a part of a number, the less of the greater, when it exactly measures the greater.

El. vii, Def. 3

The English translation cannot reproduce the word order in Euclid's Greek text. While the noun phrase 'a part of a number' (*meros arithmou*) expresses the definiendum, the verb form 'is' (*estin*) connects the definiendum with the definiens. The noun 'number' (*arithmos*), which expresses the primary subject of 'part of a number', is put after the connecting verb; it is postverbal. (In addition, also 'of a number' (*arithmou*), which is part of the definiendum, is postverbal.)

75 For example, Aristotle formulates his definition of 'time' with a preverbal position of the genus-term in certain instances (*Phys.* 220a24–25; b8–10), and with a postverbal position in another (219a34–b2).

Aristotle too takes all items in non-substantial categories—comprising relatives—to be defined by an addition of their primary subject: for instance, in defining ‘double’ and ‘half’, one respectively refers to an underlying quantity.⁷⁶ Yet, nothing indicates that Aristotle takes definitions of relatives to be formulated with a preverbal position of the primary subject term.

5.4 *Exceptions*

There seem to be some irregularities among Euclid’s definitions of the different kinds of non-substance. After showing that the preverbal formulation of the *Elements*’ definition of parallel straight lines is due to an editorial misestimation, I address the question of why Euclid’s definition of multiplication is not one of a relative.

The first kind of exception comprises only one single example. Heiberg’s edition of the *Elements* contains an additive definition of a differentia in which the primary subject term is postverbal, namely, Euclid’s infamous definition of parallel straight lines (*El.* 1, Def. 23). This seems to be a counterexample to MARK Diff_{PS-pre}. Yet, Euclid’s parallel definition of parallel planes (*El.* XI, Def. 8) has preverbal word order, in line with MARK Diff_{PS-pre}. This suggests that the word order in Def. 23 is but a meaningless aberration. While the manuscripts attest postverbal word order in Def. 23, Proclus (*in Eucl.* 175,1) still cites this definition with preverbal word order. While the manuscripts here go against Euclid’s otherwise highly regular practice of defining differentiae with a preverbal position of the primary subject term, my analysis shows that seemingly neglectable differences in Euclid’s formulations of his definitions may have crucial philosophical implications. I therefore suggest to revise Heiberg’s decision to edit Def. 23 with postverbal word order, and to replace it by the preverbal formulation recorded in Proclus.

In general, then, identifying Euclid’s markers of metaphysical distinctions proves useful, not only for reconstructing Euclid’s metaphysics, but also for the critical edition of the text of the *Elements*.

Pertaining to the second kind of exception, the *Elements* seems to contain additive definitions of a relative in which the primary subject term has a preverbal position. One might think that, if multiples of a number are relatives, then multiplication is a relative too. Euclid defines ‘multiple of a number’ by

⁷⁶ *Metaph.* 1031a2–4. In dialectical contexts, though, Aristotle draws upon the (Platonic) respondent’s view that the definition of a relative must refer to its correlative opposite, rather than to its primary subject: in defining ‘double’, one must refer to ‘half’, and vice versa: see *Top.* 142a26–31; *Sophistical Refutations* (= *SE*) 181b26–34. Still, in non-dialectical contexts, Aristotle only says that relatives are spoken of, rather than defined, by reference to their correlative opposite: see *Cat.* 6b28–36; 11b24–33; *Metaph.* 1021a26–30.

addition of the postverbal subject term 'number', and hence as a relative (by MARK Rel_{PS-post}, *El.* VII, Def. 5). Euclid's definition of 'to multiply a number' (*El.* VII, Def. 16, which is in fact Euclid's definition of multiplication) is also a definition by addition, in which case multiplication is indeed a non-substance (by MARK non-S_{Add}). However, given that the primary subject in Def. 16 is preverbal, Euclid takes 'to multiply a number' to be a differentia of the genus 'number' (by MARK Diff_{PS-pre}), instead of a relative. Therefore, multiplication is categorially different from multiples, and hence from relatives.⁷⁷

6 Conclusion

While Euclid's mathematical proofs in the *Elements* rely upon definitions, his definitions can be found to rely upon ontological classifications and metaphysical hierarchizations of mathematical objects. Euclid does not expressly formulate a philosophical theory, but his metaphysical distinctions can be reconstructed on the basis of certain linguistic encryptions in Euclid's formulations of the definitions. Euclid employs definition by division and definition by addition to distinguish substances from their non-substantial attributes. Moreover, he uses definition by division without a concrete genus-term to distinguish specific substances from generic substances, as well as definition by identification to distinguish indivisible specific substances from divisible ones. Finally, Euclid alters the position of the primary subject term within his definitions by addition to distinguish relatives from differentiae. These different types of definition are Euclid's ways of encoding traditional metaphysical distinctions.⁷⁸

⁷⁷ One might think that also 'to have a greater ratio', 'to have a duplicate ratio', and 'to have a triplicate ratio' (for magnitudes, *El.* v, Def. 7; 9–10) are Euclidean relatives, given that 'greater', 'double', and 'triple' are relative terms. However, since Euclid defines 'to have a ratio to one another' with a preverbally positioned primary subject, and hence as a differentia (by MARK Diff_{PS-pre}, *El.* v, Def. 4), and given that Def. 7 and Def. 9–10 are instances of Def. 4, their respective definiendum is a differentia. One might even regard 'number' as a relative, as does Aristotle (*Metaph.* 1092b19–20; cf. Alexander of Aphrodisias, *On Aristotle Metaphysics* 86,5–6), arguing that numbers are always numbers of something. On the other hand, Aristotle himself uses 'number' as an example of a substance term in his syllogistic (*APr* 27a19–20). At any rate, Euclid defines 'number' by division, and hence as a substance (by MARK S_{Div}, *El.* VII, Def. 2).

⁷⁸ Individual substances, however, are altogether absent in Euclid's ontology. Hintikka & Remus (1974, 5; 25; 42–44) interpret Euclid's customary practice of designating mathematical objects by letter-labels as an introduction of new individuals. Against this, Acerbi (2020) argues that Greek mathematical letter-labels are names of indefinite noun

Euclid appears to follow Aristotle's theory of definition and essence in taking only substances to serve as primary subjects of non-substantial attributes, as well as in taking the genus of a substance to specify the essence of that substance. Just like Aristotle, Euclid can thus be regarded to conceive of substance as a subject of inherence on the one hand, and as a predicate specifying the essence of the substantial subject on the other.

Euclid appears to follow Aristotle also in taking the essence of a substance to be a cause of its demonstrable non-substantial attributes, in which case Euclid's distinction of non-substantial from substantial attributes serves to distinguish demonstrable from indemonstrable attributes.

While Euclid's general metaphysical theory resembles Aristotle's corresponding theory, we have still noted quite a few divergences between Euclid's and Aristotle's respective practices of definition. Specifically, Euclid defines only substances by division, and all non-substances by an addition of their primary subject, whereas Aristotle frequently defines non-substantial attributes by division as well. Moreover, Euclid's various markers of the respective distinctions of the different subtypes of substances and non-substances are not found in Aristotle, nor in any other extant pre-Euclidean source. Aristotle defines neither generic substances by an omission of their genus, nor centre-points by an interdefinitional identification. Nor does Aristotle indicate the distinction between *differentiae* and relatives by the position of the primary subject term. Therefore, while Euclid's standards for defining substances and non-substances are most likely adopted from Aristotle, Euclid's standards for defining the various subtypes of substances and non-substances are not. This leaves room for the view that these latter standards originate with Euclid.

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phrases referring to universal mathematical objects, rather than singular terms denoting individuals.

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