In this chapter we take up Wittgenstein’s claim that the theorems of pure arithmetic are rules.

Surely, one might think, the theorems of pure arithmetic, e.g., the elementary equations have truth-conditions, and thus assertoric roles.

The truth-condition of ‘Snow is white’ is that snow is white. So also, the truth-condition of ‘7 + 5 = 12’ is that 7 + 5 = 12. The physical facts make for the truth or falsity of sentences about snow, and the mathematical facts make for the truth or falsity of sentences about 12.

But then the truth-condition of ‘7 + 6 = 12’ would be that 7 + 6 = 12, i.e., that 13 = 12. What is the sense in saying that it is a condition of the truth of anything that 13 = 12? That really is not clear to us.

One might try this: A truth-condition is that the grasp of which constitutes the understanding of a sentence.

We suppose we understand the sentence ‘7 + 6 = 12’. But what sense does it make to say that we grasp that 7 + 6 = 12 or that 13 = 12?

One might try to put it this way: To understand a sentence is to know the condition, which, were it to obtain, would render that sentence true.

Does one then know the condition which, were it to obtain, would render ‘7 + 6 = 12’ true? But there is no such condition!

To this one might respond, “Of course there is, it is precisely the condition that 7 + 6 = 12. Since you understand the sentence, you know that condition. The fact that it couldn’t obtain doesn’t mean that you don’t or can’t know it.”

This assumes that in arithmetic understanding is a matter of knowing or grasping this or that condition. But is that a correct appraisal of understanding for the sentences of arithmetic?

We understand a contradiction, and it makes no sense to speak of the conditions under which it is true. A contradiction then lacks a condition of truth. But it is not nonsense like “Twas brillig an the slithy toves did
gyre and gimble in the wabe.” In the course of an indirect proof we do not produce a piece of nonsense.

But can’t we then say that it has an unsatisfiable condition of truth, or simply say that it is associated with a condition that cannot be satisfied?

That is not so clear to us. We are not sure that we make any sense of the phrase “the condition of something being and not being a whale.”

Here is a typical response to what we are saying: “Look, the sentence ‘There are whales’ is associated with a certain condition – that condition obtains or not. The same holds for ‘There are whales which are not whales’ – it too is associated with a particular condition. One that does not obtain because it cannot obtain. But it is not as if there were no such condition.”

Some such words are often said with force and confidence. But we really are not sure that we understand them to say anything.

We understand the sentence ‘There are whales which are not whales’. But do we grasp or know the condition of there being whales that are not whales? That continues to seem very unnatural to us. What is unclear to us here is the phrase ‘the condition that there are whales which are not whales’ or the phrase ‘the condition that 12 = 13’. We find it a lot more natural to say that there are no conditions under which either of these sentences could be true. So, no condition is a truth-condition for either. They lack truth-conditions. But they can be understood.

An objection to Wittgenstein’s contention that the theorems of logic and arithmetic are rules is that these sentences are sentences with truth-conditions. A defense of Wittgenstein against this objection is this: if these sentences have truth-conditions, so do their negations; but their negations lack truth-conditions; so, therefore, do the theorems.

1. Null Sentences

Consider the following classification of truth-valued sentences. First, null sentences: sentences which say nothing whatsoever about what is or is not the case – sentences which in no way assert anything about how things are. Second, contentful sentences: sentences which say something about what is or is not the case – sentences which assert something about how things are.

1.1. Frege on Null Sentences

In section 17 of Grundlagen Frege takes up the suggestion that the theorems of logic and arithmetic are null sentences.