Is mathematics – arithmetic as well as geometry – reducible to formal logic?

Opinions differ. Kant maintained that it was not. He truly believed both geometry and arithmetic to be synthetic-apriori disciplines, heavily dependent upon our pure, non-formal intuition. Russell and Whitehead thought otherwise, and attempted to demonstrate their thesis in their *Principia Mathematica*. Afterwards Russell (1918, p. 94) triumphantly stated:¹

> The proof that all pure mathematics, including geometry, is nothing but formal logic is a fatal blow to the Kantain philosophy. (My emphasis)

But how should we understand the “pure” in “pure mathematics”?

*The Dilemma*

If “pure” means “formal”, and “formal” means “logical”, then Russell’s reductionism is sound, but trivial. If, however, “pure” means something more, i.e., something extra-logical, then this reductionism is trivially unsound.²

In the first case, the resulting reduction says very little indeed. For if that is all there is to it, then “pure chess” and “pure geography”

1. For a similar statement see Russell, 1903, pp. 4-5.
2. Russell indeed defines what he means by “pure mathematics” in the first chapter of *The Principles of Mathematics*: “Pure Mathematics is the class of all propositions of the form ‘p implies q,’ where ... neither p nor q contains any constant except logical constant” (p. 3). This definition clearly places his reductionism on the first horn of our striking dilemma.
(and the pure logical form of this paper, for that matter), are no less reducible to logic, in the same trivial sense. This being the case, were is the fatal blow to the Kantain?

The latter case would imply that “pure mathematics” and “pure geometry” are only partially formalizable to formal logic, and not fully reducible to it, as was alleged by Russell and Whitehead. For how can somthing substantially different from some other thing being fully reducible to – or, putting it differently, wholly derivable from – that other thing?

It should be remarked here that formalization and reduction are too often confused, though the difference between them is very dramatic. Formalization is commonly used in order to demonstrate the logical structure of an argument or a system. This requires jettisoning all non-formal content from the argument or the system. Reductionism, on the other hand, is commonly used to demonstrate an identity between the reduced substance and what it is being reduced to. This can be done only by preserving all (formal and non-formal) content.

Thus, if it is to be significant to the Kantain view on the synthetic-apriori status of mathematics, Russell’s reductionism must mean full and real Reductionism (rather than partial “reductionism”, viz., formalization). In other words, it must include the bold claim that all of the theorems of arithmatic and geometry – together with their meanings and functions – would be fully preserved in the reduction and not redundant. Such a complete reductionism would be anything but trivial. For, at least at first glance, arithmetical and geometrical theorems seem to have quite different meanings and usages than those of pure logic.

Reduction Vs. Formalization

Evidently, then, the question of reductionism is not: Does geometry (like geology, cosmology, astrology, chess, etc.) posses a formal, pure element that can be reduced to the standard language of logic?

3. This claim has been elaborated upon in ch. 5 of my In Defense of Metaphysics.