A SUBSTITUTIONAL FRAMEWORK
FOR ARITHMETICAL VALIDITY

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In fond memory of Nina.
(1913-1997)

1. Introduction

A platonist in mathematics believes that arithmetic has a subject matter, i.e., that the statements of arithmetic are about certain objects – the natural numbers. For a platonist, the language of (first-order) arithmetic $L_a$ is referential and he is licensed to speak of true and false sentences of $L_a$ and to endorse Tarski’s analysis of truth. It follows from this Tarskian analysis plus the fact that every natural number is denoted by some closed term of $L_a$ (a numeral, if one insists on canonicity) that the truth values of arithmetical sentences are determined by the truth values of its atomic sentences. Consider now a philosopher who, while not a platonist in any sense, broadly accepts the results of mathematics – however tentatively – and is persuaded that the truths of arithmetic are determined by the truth values of its atomic sentences (whatever may be his reformulation of the notion of arithmetical truth). This article may be viewed as an attempt to frame a position for such a non-platonist philosopher of a non-revisionist bent.

It is well known that certain atomic sentences of arithmetic have a persuasive rendering in terms of schemata of formulas of first-order languages with equality. This rendering is specially persuasive insofar as we focus on the cardinal role of numbers (and leave their ordinal role aside). For instance, the sentence $7 + 5 = 12$ can be rendered as
\(*\) \(\exists x A(x) \land \exists x B(x) \land \neg \exists x (A(x) \land B(x)) \rightarrow \exists \overline{n} (A(x) \lor B(x))\)

where \(A\) and \(B\) are any formulas of a given first-order language and where, for each numeral \(\overline{n}\), \(\exists \overline{n} x C(x)\) makes the numerical claim that there are exactly \(n\) objects \(x\) such that \(C(x)\). Such numerical claims have straightforward renderings in first-order languages with equality and, thus, expressions of the form \(\exists \overline{n} x C(x)\) are explicitly eliminable within those languages. I call a first-order scheme like \(\ast\) a checking point of arithmetic.

In the next section, I introduce a number of checking points of arithmetic, sufficient to determine arithmetical truth (in the platonic sense above). This is done in such a manner that an atomic (or negated atomic, as we will see) sentence of arithmetic is true if, and only if, the corresponding scheme consists of logically valid formulas. This feature should be enough to convince our non-revisionist philosopher that arithmetical truth (better, his reformulation thereof) is determined by logic alone. In section 3, I show how to extend the above correspondence to all sentences of first-order arithmetic. My proposal has the following general features. Given a referential first-order language with equality \(L\) enhanced with numerical quantifiers of the form \(\exists_n x\), I firstly extend \(L\) to a language \(L_{\text{sa}}\) in which substitutional quantification is permitted for the substitutional class constituted by the numerals \(\overline{n}\) occurring in the expressions \(\exists_n x\) (note that the numerals here are construed as syncategorematic, significant in context but naming nothing). Afterwards, I show how to associate with each first-order sentence \(S\) of the language of arithmetic \(L_a\) a scheme of sentences \(S\) of the substitutional language \(L_{\text{sa}}\) such that if \(S\) is true then each instance of \(S\) is logically valid. A suitable modification of the converse of this implication also holds. Hence, under this rendering, arithmetical truth is subsumed under a notion of logical validity and the nature of the determination of arithmetic by its checking points is rooted in substitutional quantification. I therefore accomplish a form of reduction of arithmetic to logic, a brand of logicism for first-order arithmetic. In the last section, I compare my substitutional approach with Gottlieb's approach as presented in [Got80]. I do not attempt to discuss the ontological issues posed by substitutional quantification. They are too intricate to be discussed in this paper. All the same, I finish