THE IRREDUCIBILITY OF THE NUMBER CONCEPT

BY

WILLEM KUYK
McGill University, Montreal

1. *Introduction*

The question whether it is possible to make some kind of ontology the basis of modern mathematics is left open by most people presently working in the mathematical fields. Fearing to introduce into mathematics arguments of a metaphysical nature, the philosophically minded mathematician will avoid as much as possible reference to mathematical existence independent of human thought. The intuitionistic attitude, for example, with respect to mathematical existence is well known. Furthermore, among those mathematicians who want to consider set theory or logic as the basis or origin of all of mathematics, the problem whether some kind of "Platonism" can be avoided of whether the greater part of mathematics is a "façon de parler", is still in discussion. In general it can be said that under the impact of the pragmatist attitude, for philosophers of mathematics the workability of mathematical systems rather than their interpretability has become a central point of view. Reflections of an epistemological nature as well as reflections regarding for example mathematical truth are not readily undertaken by mathematicians of the pragmatistic type [1].

In this paper [2] we investigate the nature and origin of the real numbers from the epistemological point of view. The motivations leading to this paper are derived from the contact the author had with the ideas of the Dutch philosopher H. Dooyeweerd, who in his work [4] insisted upon the possibility of interpreting the foundations of modern mathematics within his theory on the "modes of being". We have opinion that the view here exposed is entirely in keeping with the *attitude* of Dooyeweerd, though we do not carry through our discussion in his particular *terminology*, while we recognize that certain parts of his philosophy might need or need extension or revision (see section 7).

The basis idea of the paper is that natural numbers cannot be defined. That is to say, the concept of natural number can not be reduced to or identified with any concept that is more primitive. We will point out that the view taken here implies that real number theory is essentially a predicate

[1] For a conveniently arranged all over picture of the present state of the philosophical investigations regarding the foundations of mathematics one is referred to [1], from which readings we shall quote.

[2] This paper is a worked-out version of an address by the author to the McGill Mathematical Club in Montréal. Hence, reading the paper requires some mathematical schooling.

calculus. The advantage of looking at number theory that way is, that one obtains an interpretation of it that remains close to the intuitive motivations that guide mathematicians in forming the real number concept. We shall indicate some of the consequences of this view for set theory, though no special axiomatic approach will be suggested. It will become clear that we do not adhere to logicism. Differences also exist with intuitionism and formalism at some major points.

2. *Natural numbers as predicates*

Through the work of the early logicists Frege and Dedekind, and more specifically through the work of A. N. Whitehead and B. Russell it was shown that numbers owe their existence to classes or collections. The concept of natural number as proposed in this paper is also based upon the class concept, but in a different way. Consider first an instance of natural number, say the number two.

In order to indicate what two is we are bound to form the concept of a couple. A couple is for example the collection [a, b] consisting of the symbols a and b. Now a couple is by definition any collection that is equivalent [3] to a given couple, say [a, b]. Given this definition of couple, the number two can be circumscribed as follows: the number two is that common property of couples which makes that one recognizes any couple to be individually different from any collection that is not a couple. Generalizing this to the number of a collection, we obtain: *the number of a collection A is that common property of A and any collection that is equivalent to A, which makes that one may recognize any such collection to be individually different from any collection that is not equivalent to A.*

We finally get the following general description of number: *a number is anything that is the number of some collection.*

The following comments may clarify and explain further what is said here about number.

First, we by no means attempt to give a definition of number. In fact, saying that a predicate n (a number n) applies to a certain class or collection C means that we have a procedure to find standard class N to which n applies and which is equivalent to C. Standard classes N can be obtained by making use of a sequence of symbols I, II, II, IIII, . . . or in any other way which is usually deemed to be expedient. In calling the italicized passage above not a definition of number but a description, we concede that it is impossible to grip completely that, which in the plastic and time-dependent horizon of our human experience is recognized as the "number of a collection". In trying to find a satisfactory definition of number we came to the conclusion that those definitions which use a terminology like "the class of all things with . . ." are deficient in the sense that they either lead to circularities of one kind or another, or violate the predicative nature of number. Such a terminology, moreover, presupposes an a priori philosophical class or set theory.

In the second place, we would point out, by comparison with the one given by B. Russell, that the above number concept lacks the logicist touch.

[3] Two collections are equivalent, if one can bring the elements of the one collection into a one-to-one correspondence with the elements of the other collection.