Some early Greek attempts to square the circle

A. WASSERSTEIN

The quadrature of the circle, one of the most famous mathematical problems of antiquity no less than of more modern times, appears comparatively early in the history of Greek mathematics. In the 5th century it had an established position as one of the three traditional problems, viz. quadrature or rectification of the circle, duplication of the cube and trisection of a given angle. Thinkers as different from each other as Hippocrates of Chios¹ Anaxagoras the philosopher² and Antiphon the Sophist³ worked on it; and by 414 B.C. the endeavour "to square the circle" had become the recognized sign of the crank.⁴

In what follows I shall (a) examine the evidence that we have for some of the reported attempts to square the circle, and (b) attempt to reconstruct the historical relation that subsists between these attempts.

Aristotle mentions an attempt by Antiphon the Sophist to square the circle.⁵—

ἀμα δ', οὖν ἠλευν ἀπαντα προσῆκε, ἀλλ' ἐκ τῶν ἄρχων τις ἐπιδεικνύος ψεύδεται, δι' ἔκ τῶν τετραγωνισμῶν τὸν μὲν διὰ τῶν τετραγώνων γεωμετρίκου διαλύσατι, τὸν δὲ 'Ἀντιφόντος οὖ γεωμετρίκου.

"No man of science is bound to solve every kind of difficulty that may be raised, but only as many as are drawn falsely from the principles of the science: it is not our business to refute those that do not arise in this way: just as it is the duty of the geometer to refute the squaring of the circle by means of segments, but it is not his duty to refute Antiphon's proof." (Translation by R. P. Hardie in the Oxford translation of Aristotle.)

We owe our knowledge, such as it is, of Antiphon's method to the commentators on the Physics.⁶

¹ For Hippocrates' quadrature of lunes see Arist. Phys. I. 2. 185a 16 and Simpl. 55, 26; 60, 22 and 61. 5-68, 32; Themistius in Phys. p. 3 Schenkl; Procl. in Eucl. I. p. 66 Friedlein; Philop. in Phys. 31, 3; Olympiod. (in Arist. Meteorol.) 45. 24. He probably knew already that "plane" methods were not sufficient for a solution of the problem of squaring the circle. But it is precisely this that shows how far 5th century mathematicians had gone in their thinking about the problem.

² Plut. de exil. 17, p. 607 F.

³ See below.

⁴ Cf. Aristophanes Birds, 1005; the play was produced in 414 B.C.

⁵ Phys. I. 2. 185a 14-17.


92
Themistius, in Phys. p. 4, 2 sqq. says

Πρὸς Ἀντιφώντα δὲ οὐκὲν ἄν ἔχει λέγειν ὁ γεωμέτρης, δι᾽ ἐγγράφων τριγώνων ἑσπερευτῶν εἰς τὸν κύκλον καὶ ἐφ᾽ ἐκάστης τῶν πλευρῶν ἔτερον ἰσοσκελὲς συνιστάς πρὸς τῇ περιφερείᾳ τοῦ κύκλου καὶ τοῦτο ἐφεξῆς ποὺών ἕστω ποτὲ ἑραμόσειν τοῦ τελευταίου τριγώνου τὴν πλευρὰν ἐφθείναν οὐσαν τῇ περιφερείᾳ. τοῦτο δὲ τῇ ἐπ᾽ ἀπειρον τομῇ ἀναιροῦντος, ἦν ὑπόθεσιν ὁ γεωμέτρης λαμβάνει.


Antiphon drew a circle and inscribed in it a polygonal figure of the sort that can be inscribed (in a circle). Let us assume that the inscribed figure is a square. Then dividing each of the sides of the square into two halves, he drew perpendiculars from the points of division to the circumference; these perpendiculars obviously divided into halves the corresponding sections of (the circumference of) the circle. Then he drew straight lines from the points (at which the perpendiculars cut the circumference) to the extremities of the sides of the square, so as to get four triangles based on the straight lines (i.e. the sides); and the whole (resulting) inscribed figure will now be an octagon. And this process he repeated in the same way, dividing each of the sides of the octagon into halves, drawing perpendiculars from the points of division to the circumference and connecting by straight lines the points at which the perpendiculars met the circumference with the extremities of the divided straight lines (i.e. the sides of the octagon); the inscribed figure he thus obtained was a polygon with sixteen sides. In the same way cutting the sides of the inscribed sixteensided polygon and drawing straight connecting lines he doubled (the number of the sides of) the inscribed polygon; and repeating the process again and again (ἀξιό) so that in this way, with the progressive 1 exhaustion of the area (of the circle) a polygon would be inscribed in the circle, the sides of which because of their smallness would coincide with the circumference of the circle. Since we can always draw a square equal to any (given) polygon, as we learn in the Elements,2 we shall in fact be constructing a square equal to a circle, seeing that a polygon coinciding with a circle is equal to it.

It is evident that this conclusion infringes geometrical principles; but not, as Alexander said, “because the geometer assumes as a principle that the circle touches the straight line at one point, and that is the principle infringed by Antiphon”; for the geometer does not assume this but demonstrates it, in the third book.3 It is therefore better to say that the principle is this: that it is impossible for a straight line to coincide with a circumference. A line outside (the circle) will touch it at one point only; a line inside at two points and no more; and contact takes place at a point. The progressive division of the area between the straight line and the circumference of the circle will not exhaust it, nor will one ever reach the circumference, if (as is the case) the area is divisible ad infinitum. If one does reach the circumference that would infringe the geometrical principle which states that magnitudes are divisible ad infinitum. And it is this principle that according to Eudemus was infringed by Antiphon.

1 I think I am justified in introducing this word here; notice the fact that διαπεραιμένου is a present participle.
2 See Euclid II.14.
3 See Euclid III.16.